

**SCALE INVARIANT  $O(g^4)$  LIPATOV KERNELS  
AT NON-ZERO MOMENTUM TRANSFER\***Claudio Corianò,<sup>a,b†</sup> Rajesh R. Parwani<sup>c+</sup> and Alan. R. White<sup>a#</sup><sup>a</sup>High Energy Physics Division  
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Bhubaneswar 751005, India.**Abstract**

We summarize recent work on the evaluation of the scale invariant next-to-leading order Lipatov kernel, constructed via transverse momentum diagrams. At zero momentum transfer the square of the leading-order kernel appears together with an additional component, now identified as a new partial-wave amplitude, having a separate, holomorphically factorizable, spectrum. We present a simplified expression for the full kernel at non-zero momentum transfer and give a complete analysis of its infrared properties. We also construct a non-forward extension of the new amplitude which is infra-red finite and satisfies Ward identity constraints. We conjecture that this new kernel has the conformal invariance properties corresponding to the holomorphic factorization of the forward spectrum.

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## 1. A BRIEF OVERVIEW

The Regge limit of QCD has recently undergone a considerable revival of interest. The small- $x$  behaviour of the parton distributions observed at HERA, characterized by a strong rise of the gluon density, and the detection of diffractive hard scattering events in DIS, both provide motivation for developing a better understanding of the Regge regime of QCD. Because of the overlap of the small- $x$  and Regge limits, it is natural to expect that the theoretical tools developed in the past in the analysis of Regge theory are useful also at small- $x$ . Properties of the “exchanged reggeon singularities” can be constructed from perturbation theory, by resumming the leading  $\log 1/x$  and/or  $\log Q^2$  behaviour. Resummation is achieved in various possible ways, but it is widely anticipated that the BFKL evolution equation [1], first derived more than 20 years ago, plays a crucial role in describing the physical properties of the leading “Pomeron” singularity.

The crucial ingredient in the “construction” of the BFKL Pomeron is the kernel of the evolution equation, its spectrum and its leading eigenvalue. Both forward ( $q = 0$ ) and non-forward ( $q \neq 0$ ) versions of the lowest order ( $O(g^2)$ ) kernel are known. Conformal partial waves diagonalize the  $O(g^2)$  equation at non zero  $q$ , since the equation is invariant under special conformal transformations, and in the limit of  $q \rightarrow 0$  reproduce the well known eigenfunctions, and eigenvalues of the BFKL *parton* (or forward) kernel. A necessary condition for the conformal invariance of the equation is the property of holomorphic factorization of the eigenvalues of the parton kernel.

Most analyses of the BFKL equation involve only the  $O(g^2)$  kernel and its related properties of conformal invariance. It is, of course, important to see how radiative corrections affect the leading order evolution. It is expected that renormalization effects will introduce a running of the coupling and will spoil conformal invariance. The direct evaluation of next-to-leading-order(NLO) contributions to this equation requires both a calculation of the correction to the Regge trajectory of the gluon and corresponding corrections to the reggeon(s)-particle(s) transition vertices. So far only part of this program has been completed[2].

Both the leading-order kernel and an infrared approximation to the NLO order kernel have been determined by a reggeon diagram technique based indirectly on  $t$ -channel unitarity[3]. More recently we have shown[4, 5] how these results can be

obtained by a direct analysis of the  $t$ -channel unitarity equations, analytically continued in the complex  $j$  – *plane* and expanded around nonsense poles. Also, in a recent paper, Kirschner[6] has discussed how the same NLO kernel may emerge as an approximation when non-leading results are obtained using the  $s$ -channel multi-Regge effective action. The kernel obtained is automatically scale invariant (there is no scale in the  $t$ -channel analysis) and is naturally expressed in terms of two-dimensional transverse momentum integrals. We should emphasize, nevertheless, that both the  $t$ -channel analysis[5] and the  $s$ -channel formalism[6], imply that the ambiguity of the scale-dependence includes the overall normalization of the kernel.

Previously we have shown[7] that, in the forward case, the new NLO kernel splits naturally into two components. A part proportional to the square of the  $O(g^2)$  kernel and a new component that is separately infrared safe and has an eigenvalue spectrum sharing many of properties of the leading-order spectrum. In particular the very important property of holomorphic factorization. From the unitarity analysis[5] we have now shown that this new component is actually *a distinct partial-wave amplitude that appears for the first time at  $O(g^4)$* . It is natural to expect that the spectral property of holomorphic factorization will be related to the leading-order conformal invariance of this amplitude when it is fully identified as a non-forward kernel.

In this work we are going to elaborate further on the non-forward, scale invariant NLO kernel, by providing an explicit proof of its infrared safety and simplifying drastically the expression given in [7]. For this purpose we extend to the non-forward case a method of calculation of the various diagrams, based on the use of complex momenta, due to Kirschner[6]. This method has been successful in reproducing the spectrum calculated in [7] and in separating, in the forward direction, the new holomorphically factorizable component. We will explicitly construct a non-forward extension of this component that has the appropriate analytic structure and satisfies the Ward identity constraints. We believe that this extension can indeed be identified as a new partial-wave amplitude which at “leading-order” is conformally invariant. We intend to study this issue in the near future. (Note that since the spectrum of a conformally invariant kernel is independent of  $q^2$  it is, in principle, defined uniquely by the forward spectrum.)

We will also show that the new non-forward (potentially conformally invariant) kernel is not naturally written as a transverse momentum integral but rather is simply expressed in the complex momenta formalism of Kirschner. This is interesting because the unitarity formalism of [5] actually shows that *the transverse momentum integral*

formalism is only necessarily applicable, as  $q^2 \rightarrow 0$ , and for the leading threshold behaviour in reggeon mass variables. We show explicitly that extracting this threshold behavior from the transverse momentum integral kernel is not sufficient to give the desired non-forward extension. It is only at  $q^2 = 0$  that a transverse momentum integral gives the appropriate threshold behavior.

## 2. THE FORWARD KERNEL

Consider first the leading-order BFKL evolution equation for parton distributions at small- $x$  i.e.

$$\frac{\partial}{\partial(\ln 1/x)} F(x, k^2) = \tilde{F}(x, k^2) + \frac{1}{(2\pi)^3} \int \frac{d^2 k'}{(k')^4} K(k, k') F(x, k'^2) \quad (2.1)$$

with a parton kernel  $K(k, k')$  given (for  $SU(N)$ ) by

$$(Ng^2)^{-1} K(k, q) = \left( -\delta^2(k - k') k^6 \int \frac{d^2 p}{p^2 (k - p)^2} + \frac{2k^2 k'^2}{(k - k')^2} \right) \quad (2.2)$$

The original Regge limit derivation included also a non-forward (i.e.  $q \neq 0$  in the following) version of this equation. Transforming to  $\omega$  - space, where  $\omega$  is conjugate to  $\ln \frac{1}{x}$ , the non-forward equation takes the form

$$\omega F(\omega, k, q - k) = \tilde{F} + \frac{1}{16\pi^3} \int \frac{d^2 k'}{(k')^2 (k' - q)^2} K(k, k', q) F(\omega, k', q - k') \quad (2.3)$$

where the “reggeon” kernel  $K(k, k', q) = K_{2,2}^{(2)}(q - k, k, k', q - k')$  contains three kinematic forms.

$$\begin{aligned} \frac{1}{Ng^2} K_{2,2}^{(2)}(k_1, k_2, k_3, k_4) = & \sum \left( -\frac{1}{2} k_1^4 J_1(k_1^2) k_2^2 (16\pi^3) \delta^2(k_2 - k_3) \right. \\ & \left. + \frac{k_1^2 k_3^2}{(k_1 - k_4)^2} - \frac{1}{2} (k_1 + k_2)^2 \right) \equiv K_1^{(2)} + K_2^{(2)} + K_3^{(2)} . \end{aligned} \quad (2.4)$$

where

$$J_1(k^2) = \frac{1}{16\pi^3} \int \frac{d^2 k'}{(k')^2 (k' - k)^2} \quad (2.5)$$

and the  $\sum$  implies that we sum over *combined* permutations of both the initial and the final state (i.e.  $1 < - > 2$  combined with  $3 < - > 4$ ).

We use transverse momentum diagrams, which we construct using the components illustrated in Fig. 2.1.



Fig. 2.1 (a)vertices and (b) intermediate states in transverse momentum.

The rules for writing amplitudes corresponding to the diagrams are the following

- For each vertex, illustrated in Fig. 2.1(a), we write a factor

$$16\pi^3\delta^2(\sum k_i - \sum k'_i)(\sum k_i)^2$$

- For each intermediate state, illustrated in Fig. 2.1(b), we write a factor

$$(16\pi^3)^{-n} \int d^2k_1 \dots d^2k_n / k_1^2 \dots k_n^2$$

Dimensionless kernels are defined by a hat

$$\hat{K}_{2,2}^{(2)}(k_1, k_2, k_3, k_4) = 16\pi^3\delta^2(k_1 + k_2 - k_3 - k_4)K_{2,2}^{(2)}(k_1, k_2, k_3, k_4)$$

The kernels so defined are formally scale-invariant (even though potentially infra-red divergent). The diagrammatic representation of  $\hat{K}_{2,2}^{(2)}$ , the non forward BFKL kernel, is then as in Fig. 2.2.

$$\sum \left( -\frac{1}{2} \text{---}\bigcirc\text{---} + \text{---}\diagdown\diagup\text{---} - \frac{1}{2} \text{---}\times\text{---} \right)$$

Fig. 2.2 Diagrammatic representation of  $\hat{K}_{2,2}^{(2)}$

The summation sign again implies a sum over combined permutations of the initial and final momenta.

The  $O(g^4)$  transverse momentum integral kernel  $K_{2,2}^{(4)}$ , obtained by considering the contribution of the 4-particle nonsense states to the unitarity equations is defined by the sum

$$\frac{1}{(g^2 N)^2} K_{2,2}^{(4n)}(k_1, k_2, k_3, k_4) = K_0^{(4)} + K_1^{(4)} + K_2^{(4)} + K_3^{(4)} + K_4^{(4)} . \quad (2.6)$$

with

$$K_0^{(4)} = \sum k_1^4 k_2^4 J_1(k_1^2) J_1(k_2^2) (16\pi^3) \delta^2(k_2 - k_3) , \quad (2.7)$$

$$K_1^{(4)} = -\frac{2}{3} \sum k_1^4 J_2(k_1^2) k_2^2 (16\pi^3) \delta^2(k_2 - k_3) \quad (2.8)$$

$$K_2^{(4)} = -\sum \left( \frac{k_1^2 J_1(k_1^2) k_2^2 k_3^2 + k_1^2 k_3^2 J_1(k_4^2) k_4^2}{(k_1 - k_4)^2} \right), \quad (2.9)$$

$$K_3^{(4)} = \sum k_2^2 k_4^2 J_1((k_1 - k_4)^2) , \quad (2.10)$$

and

$$K_4^{(4)} = \frac{1}{2} \sum k_1^2 k_2^2 k_3^2 k_4^2 I(k_1, k_2, k_3, k_4), \quad (2.11)$$

where  $J_1(k^2)$  is defined by (2.5) and

$$J_2(k^2) = \frac{1}{16\pi^3} \int d^2 q \frac{1}{(k - q)^2} J_1(q^2) , \quad (2.12)$$

and

$$I(k_1, k_2, k_3, k_4) = \frac{1}{16\pi^3} \int d^2 p \frac{1}{p^2 (p + k_1)^2 (p + k_1 - k_4)^2 (p + k_3)^2}. \quad (2.13)$$

The diagrammatic representation of  $\hat{K}_{2,2}^{4n}$  is shown in Fig. 2.3.

$$\Sigma \frac{1}{2} \left( \text{diagram 1} - \frac{2}{3} \text{diagram 2} - \text{diagram 3} - \text{diagram 4} + \text{diagram 5} + \frac{1}{2} \text{diagram 6} \right)$$

Fig. 2.3 The diagrammatic representation of  $\hat{K}_{2,2}^{4n}$ .

The evaluation of these diagrams (in particular the non planar box) has been done by an extension of the Källen and Toll method [8], developed in [7]. This involves a rewriting of the “trees” [8] of the decomposition in a suitable base. The decomposition has the advantage of generating a minimal number of logarithms. The proliferation of logarithms at NLO is a considerable source of complexity. (At leading order there is a logarithm only in the trajectory function of the gluon.) In particular, the box introduces 6 logarithms, each of which is obtained by putting on shell 2 lines (pairwise) and which we represent as in Fig. 2.4.

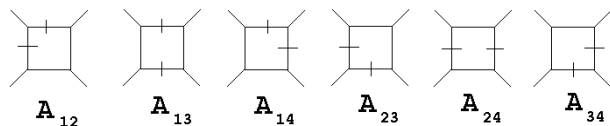


Fig. 2.4 Tree diagrams obtained by putting on-shell the crossed lines.

The logarithms are of two types:

- 1) external line “mass” thresholds i.e.  $A_{12}, A_{14}, A_{23}$  and  $A_{34}$  - four logarithms.
- 2) “s” and “t” thresholds i.e.  $A_{13}, A_{24}$  - two logarithms.

In the forward direction it is straightforward to combine the type 2) logarithms from the box with the logarithms of the connected components  $K_2^{(4)}$  and  $K_3^{(4)}$  giving (in the notation of [7]  $\mathcal{K}_1$ ). Adding the logarithms generated by the disconnected components  $K_0^{(4)}, K_1^{(4)}$  (denoted in [7] as  $\mathcal{K}_0$ ) gives a kernel which is infrared safe both before and after convolution with the eigenfunctions and is equal to the *square* of the lowest order BFKL kernel  $K_{2,2}^{(2)}$ . That is we have the identity

$$\hat{\mathcal{K}}_0 + \hat{\mathcal{K}}_1 = \frac{1}{4} \left( \hat{K}_{2,2}^{(2)} \right)^2, \quad (2.14)$$

The proof of this identity is given in [7].

The set of box diagram logarithms 1) was denoted in [7] as  $\mathcal{K}_2$ . It contains only the mass thresholds and is the contribution which, in the forward direction, the unitarity analysis of [5] determines should be correctly given by the transverse momentum integral formalism. It is a new, separately infrared finite, kernel for which

the spectrum has been calculated and shown to satisfy the property of holomorphic factorization[7]. Therefore for the full forward, or parton, kernel we can write

$$K_{2,2}^{(4)} = g^2 K_{BFKL} + O(g^4) \left[ (K_{BFKL}/2)^2 + \mathcal{K}_2 \right]. \quad (2.15)$$

where both  $K_{BFKL}$  and  $\mathcal{K}_2$  have a spectrum which is holomorphically factorizable. In both cases the spectrum is also infrared(IR) safe. (In writing “ $O(g^4)$ ” in (2.15) we have indicated the normalization uncertainty due to scale dependence.)

While a direct check of IR safety is easily accomplished in the case of the forward kernel, the case of the non forward kernel, starting from its explicit expression given in [7], is far less obvious. The proof of infrared safety given there involves diagrammatic identities. In the next sections we are going to reproduce this cancellation by defining suitable, consistent, regularizations of the various components of the kernel. We will work directly in two dimensions and show from the final expression that the resulting kernel is IR finite. The method employs an analytic continuation of the diagrams to complex space.

### 3. THE NON-FORWARD TRANSVERSE MOMENTUM DIAGRAM KERNEL

Kirschner has recently shown [6] that the same separation of the  $\mathcal{K}_2$  component from the remaining part of the  $g^4$  kernel, first obtained in [7], can be reobtained by performing a complex expansion of the relevant diagrams. Here we extend his method of calculation to the non-forward case. We complexify the “propagators” and “vertices” as follows. We write

$$\begin{aligned} \frac{1}{k^2} &\rightarrow \frac{1}{kk^*} \equiv \frac{1}{|k|^2} \\ (k+q)^2(k-q)^2 &\rightarrow |k+q|^2|k-q|^2 = |k^2 - q^2|^2 |k'^2 - q^2|^2. \end{aligned} \quad (3.1)$$

That is we replace all the momenta  $k = (k_0, k_1)$  by their complex versions  $k = k_0 + ik_1$ . We also define

$$RR' \equiv |k^2 - q^2|^2 |k'^2 - q^2|^2 \quad (3.2)$$

The contribution of the box diagram to  $K_4^{(4)}$  is now given by

$$I[box] = RR' \int \frac{d^2 l}{|(l+k+q)|^2 |l|^2 |(l+k+k')|^2 |(l+k'+q)|^2}. \quad (3.3)$$



We partial fraction the denominator by writing

$$A = \frac{1}{l(l+k'+k)} = \frac{1}{(k+k')} \left( \frac{1}{l} - \frac{1}{l+k+k'} \right)$$

$$B = \frac{1}{(l+k+q)(l+k'+q)} = \frac{1}{(k-k')} \left( \frac{1}{l+k+q} - \frac{1}{l+k'+q} \right),$$

so that

$$I[box] = RR' \int \frac{dldl^*}{4|q|^2|l+\eta|^2} (|A|^2 + |B|^2 + A^*B + AB^*) \quad (3.4)$$

where

$$\eta \equiv \frac{(k+q)(k'+q)}{2q} \quad (3.5)$$

In the limit  $q \rightarrow 0$  one can show that the “mixed products”  $A^*B$  and  $AB^*$  give directly that part of the box which we have identified above as  $\mathcal{K}_2$ . As we have discussed previously and will discuss further below, there are good reasons to think this part of the scale invariant kernel is a new contribution at NLO which is not related to renormalization effects.

The partial fractioning technique that is the basis of our analysis introduces spurious singularities and we need to introduce regulators in order to give meaning to the complex two-dimensional integrals involved. We will define  $\int_{\Lambda_1}$ ,  $\int_{\mu_1}$  and  $\int_{\Lambda_1, \mu_1}$  to be suitable UV, IR and UV-IR regularizations of the corresponding integrals by *defining*

$$\int_{\Lambda_1} \frac{dldl^*}{l(l+\eta)^*} = 2\pi \log \frac{\Lambda_1}{|\eta|}$$

$$\int_{\Lambda_1 \mu_1} \frac{dldl^*}{|l+\eta|^2} = 2\pi \log \frac{\Lambda_1}{\mu_1}. \quad (3.6)$$

The second integral is discussed further in Appendix A. It is easy to show that all the spurious UV singularities introduced by the complex decomposition cancel. The infrared singularities, instead, in single integrals which are IR divergent, do not cancel. The analysis of their cancellation is the non trivial part of our analysis. With the above definitions we obtain e.g.

$$I_1 = \int_{\mu_1} \frac{dldl^*}{|l+\eta|^2 l(l+k'+q)^*} = \frac{2\pi}{\eta(k'+q-\eta)^*} \log \frac{|k'+q|\mu_1}{|k'+q-\eta||\eta|} \quad (3.7)$$

and

$$\int_{\mu_1} \frac{dldl^*}{|l + \eta|^2(l + k + k')(l + k + q)^*} = \frac{2\pi}{\eta'^*(k' - q - \eta')} \log \frac{|k' - q|\mu_1}{|\eta'| |k' - q - \eta|}, \quad (3.8)$$

where  $\eta' \equiv \eta - k - q$ .

We can now evaluate the integrals involving  $A$  and  $B$  as follows. The “mixed” terms give

$$\begin{aligned} I[AB^*] + c.c. &= |k^2 - q^2|^2 |k'^2 - q^2|^2 \int \frac{dldl^* AB^*}{|2ql + (k + q)(k' + q)|^2} \\ &= \frac{|k^2 - q^2|^2 |k'^2 - q^2|^2}{4|q|^2(k + k')(k - k')^*} \left( \frac{2\pi}{\eta(k + q - \eta)^*} \log \frac{|k + q|\mu_1}{|\eta| |k + q - \eta|} \right. \\ &\quad - \frac{2\pi}{\eta(k' + q - \eta)^*} \log \frac{|k' + q|\mu_1}{|\eta| |k' + q - \eta|} - \frac{2\pi}{(\eta - k - q)^*(k + k' - \eta)} \log \frac{|k' - q|\mu_1}{|\eta - k - q| |k + k' - \eta|} \\ &\quad \left. + \frac{2\pi}{(\eta - k' - q)^*(k + k' - \eta)} \log \frac{|k - q|\mu_1}{|\eta - k' - q| |k + k' - \eta|} \right) + c.c. \end{aligned} \quad (3.9)$$

Similarly we obtain

$$\begin{aligned} I[|A|^2] &= |k^2 - q^2|^2 |k'^2 - q^2|^2 \int \frac{dldl^* |A|^2}{|2ql + (k + q)(k' + q)|^2} \\ &= \frac{|k^2 - q^2|^2 |k'^2 - q^2|^2}{4|q|^2 |k + k'|^2} \left( -\frac{2\pi}{\eta(k + k' - \eta)^*} \log \frac{|k + k'|\mu_1}{\eta(k + k' - \eta)} - \frac{2\pi}{\eta^*(k + k' - \eta)} \right. \\ &\quad \left. \times \log \frac{|k + k'|\mu_1}{|\eta| |k + k' - \eta|} + \frac{4\pi}{|\eta|^2} \log \frac{|\eta|}{\mu_1} + \frac{4\pi}{|k + k' - \eta|^2} \log \frac{|k + k' - \eta|}{\mu_1} \right) \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} I[|B|^2] &= |k^2 - q^2|^2 |k'^2 - q^2|^2 \int \frac{dldl^* |B|^2}{|2ql + (k + q)(k' + q)|^2} \\ &= \frac{2|k^2 - q^2|^2 |k'^2 - q^2|^2}{4|q|^2 |k - k'|^2} \left( \frac{2\pi}{|k + q - \eta|^2} \log \frac{|k + q - \eta|}{\mu_1} \right. \\ &\quad \left. + \frac{2\pi}{|k' + q - \eta|^2} \log \frac{|k' + q - \eta|}{\mu_1} - \frac{\pi}{(k + q - \eta)^*(k' + q - \eta)} \log \frac{|k' + q - \eta| |k + q - \eta|}{\mu_1 |k' - k|} \right) \end{aligned}$$

$$-\frac{\pi}{(k+q-\eta)^*(k'+q-\eta)}\log\frac{|k'+q-\eta||k+q-\eta|}{\mu_1|k'-k|} \quad (3.11)$$

Moving on to the other connected components of  $K_{2,2}^{(4)}$ , we obtain

$$\begin{aligned} K_2^{(4)}(k, k', q) = & - \left( \frac{4\pi|k+q|^2|k'+q|^2}{|k+k'|^2}\log\frac{|k'-q|^2}{\mu_1} + \frac{4\pi|-k+q|^2|k'+q|^2}{|-k+k'|^2}\log\frac{|k'-q|^2}{\mu_1} \right. \\ & + \frac{4\pi|k+q|^2|-k'+q|^2}{|k-k'|^2}\log\frac{|k'-q|^2}{\mu_1} + \frac{4\pi|-k+q|^2|-k'+q|^2}{|k+k'|^2}\log\frac{|k'+q|^2}{\mu_1} \\ & + \frac{4\pi|k-q|^2|k'-q|^2}{|k+k'|^2}\log\frac{|k+q|^2}{\mu_1} + \frac{4\pi|k+q|^2|k'-q|^2}{|-k+k'|^2}\log\frac{|-k+q|^2}{\mu_1} \\ & \left. + \frac{4\pi|k-q|^2|k'+q|^2}{|k-k'|^2}\log\frac{|k+q|^2}{\mu_1} + \frac{4\pi|k+q|^2|k'+q|^2}{|k+k'|^2}\log\frac{|k-q|^2}{\mu_1} \right) \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} K_3^{(4)}(k, k', q) = & \left( \frac{4\pi|k+q|^2|k'+q|^2}{|k+k'|^2}\log\frac{|k+k'|^2}{\mu_1} + \frac{4\pi|-k+q|^2|k'+q|^2}{|-k+k'|^2}\log\frac{|-k+k'|^2}{\mu_1} \right. \\ & \left. + \frac{4\pi|k+q|^2|-k'+q|^2}{|k-k'|^2}\log\frac{|k-k'|^2}{\mu_1} + \frac{4\pi|-k+q|^2|-k'+q|^2}{|k+k'|^2}\log\frac{|k+k'|^2}{\mu_1} \right) \end{aligned} \quad (3.13)$$

## 4. INFRARED CANCELLATIONS

In order to prove that the complete kernel,  $K_{2,2}^{(4)}$ , is IR safe, and to simplify the notation, let's define  $\mathcal{R}_{\mu_1}$  as the operation which isolates the infrared sensitive logarithms of all the components i.e. the coefficient of  $\log\mu_1$ . We get (omitting an overall factor of  $2\pi$ )

$$\mathcal{R}_{\mu_1} * I[AB^*] = -\frac{RR'}{4|q|^2} \left( \frac{1}{(k+q-\eta)(k'+q-\eta)^*\eta^*(k+k'-\eta)^*} + c.c. \right) \quad (4.1)$$

leading to

$$\mathcal{R}_{\mu_1} * I[AB^* + A^*B] = 8|q|^2, \quad (4.2)$$

This shows that the "mixed" contributions are separately IR safe only in the forward direction i.e.  $q = 0$ .

We similarly obtain

$$\mathcal{R}_{\mu_1} * I[|A|^2] = -4|q|^2 - \frac{|k - q|^2|k' - q|^2 + |k + q|^2|k' + q|^2}{|k + k'|^2} \quad (4.3)$$

and

$$\mathcal{R}_{\mu_1} * I[|B|^2] = -4|q|^2 - \frac{|k + q|^2|k' - q|^2 + |k - q|^2|k' + q|^2}{|k + k'|^2} \quad (4.4)$$

Combining these last results with (4.2) we see that  $R_{\mu_1} * I[box]$  is non zero. We conclude that the box diagram is not separately IR safe.

We also obtain

$$\begin{aligned} \mathcal{R}_{\mu_1} * K_4^{(4)} &= -\frac{2}{|k + k'|^2} \left( |k - q|^2|k' - q|^2 + |k + q|^2|k' + q|^2 \right) \\ &\quad - \frac{2}{|k - k'|^2} \left( |k - q|^2|k' + q|^2 + |k + q|^2|k' - q|^2 \right) \end{aligned} \quad (4.5)$$

and, after a quite involved pattern of cancellations,

$$\begin{aligned} \mathcal{R}_{\mu_1} * (K_2^{(4)} + K_3^{(4)}) &= +\frac{2}{|k + k'|^2} (|k' + q|^2|k - q|^2 + |k' - q|^2|k + q|^2) \\ &\quad + \frac{2}{|k - k'|^2} (|k' + q|^2|k - q|^2 + |k' - q|^2|k + q|^2) \end{aligned} \quad (4.6)$$

Combining (4.2), (4.3), (4.4), (4.5) and (4.6) we obtain

$$R_{\mu_1} * (K_2^{(4)} + K_3^{(4)} + K_4^{(4)}) = 0 \quad (4.7)$$

showing that the infra-red divergences cancel.

## 5. SEPARATION OF THRESHOLDS

The holomorphic factorization properties of  $\mathcal{K}_2$  clearly suggest that there should be a conformally invariant extension to the non-forward direction. Since we

expect to identify this extension as a new partial-wave reggeon amplitude we look for a separately infra-red finite component of the non-forward kernel which satisfies the Ward identity constraint that it vanish when any  $k_i \rightarrow 0$ ,  $i = 1, \dots, 4$ . From the unitarity analysis of [5] and the discussion in [7] we know that we should try to isolate the thresholds from the box diagram associated with logarithms of type 1) discussed in Section 2.

The logarithms we are interested in are again present in the mixed terms  $AB^*$  and  $A^*B$  discussed in the last Section. However, there are also additional logarithms of the form  $q^2 \log 4q^2$ , which are associated with infra-red divergences that appear. If we extract these logarithms we obtain

$$\begin{aligned}
\mathcal{I}_{AB}(q) &= -(1 - R_q - R_{\mu_1}) * I[AB^* + c.c.] \\
&= \frac{2\pi(k+q)^2(k-q)^2(k'+q)^2(k'-q)^2}{(k+k')^2(k-k')^2} \\
&\times \left( \frac{(k'^2 - q^2)(k^2 - k'^2) + [qk'][k'k]}{(k+q)^2(k'-q)^2(k'+q)^2} \log(k'+q)^2(k'-q)^2(k+q)^2 \right. \\
&\quad - \frac{(k^2 - q^2)(k^2 - k'^2) + [qk][k'k]}{(k'+q)^2(k-q)^2(k+q)^2} \log(k+q)^2(k-q)^2(k'+q)^2 \\
&\quad - \frac{(k^2 - q^2)(k^2 - k'^2) + [kq][k'k]}{(k+q)^2(k'-q)^2(k-q)^2} \log(k+q)^2(k-q)^2(k'-q)^2 \\
&\quad \left. + \frac{(k'^2 - q^2)(k^2 - k'^2) + [k'q][k'k]}{(k'+q)^2(k-q)^2(k'-q)^2} \log(k'+q)^2(k'-q)^2(k-q)^2 \right).
\end{aligned} \tag{5.1}$$

where we have defined  $[qk] \equiv q^* k - q k^*$ . Introducing vectors  $\hat{k} = (k_1, -k_0)$ , dual to  $k = (k_0, k_1)$ , with the properties  $\hat{k}^2 = k^2$  and  $\hat{k} \cdot k = 0$ ,

$$[qk] = 2i\hat{k} \cdot q. \tag{5.2}$$

It is straightforward to check that

$$\begin{aligned}
\mathcal{I}_{AB}(q) &\xrightarrow{q^2 \rightarrow 0} \frac{(k^2 - k'^2)k^2 k'^2}{(k+k')^2(k-k')^2} \log \frac{k^2}{k'^2} \\
&= \mathcal{K}_2
\end{aligned} \tag{5.3}$$

However,  $\mathcal{I}_{AB}(q)$  has several problems if we wish to identify it as a non-forward extension of  $\mathcal{K}_2$ . It is not infra-red finite in the sense that the arguments of the

logarithms are not ratios of momentum factors. In addition the behaviour at the thresholds i.e. at  $q \pm k \rightarrow 0$  and  $q \pm k' \rightarrow 0$  is sufficiently singular that *the Ward identities are not satisfied*. That is  $\mathcal{I}_{AB}(q)$  does not vanish in these limits.

We conclude that the transverse momentum integral corresponding to the non-forward box diagram does not contain the extension of  $\mathcal{K}_2$  that we are seeking. Given the limitations of the transverse momentum integral formalism away from  $q^2 = 0$  that we have discussed in [5] this is, perhaps, not surprising.

For completeness we also give here the explicit expression for the remainder of the  $O(g^4)$  connected part of  $K_{2,2}^{(4)}$ . That is if we write

$$K_2^{(4)} + K_3^{(4)} + K_4^{(4)} = \mathcal{R}(q) + \mathcal{I}_{AB}(q) \quad (5.4)$$

then

$$\begin{aligned} \mathcal{R}(q) = & \frac{2\pi(k+q)^2(k-q)^2(k'+q)^2(k'-q)^2}{(k+k')^2} \\ & \left( \frac{(k'^2 - q^2)(k^2 - q^2) + 2[qk][qk']}{(k+q)^2(k-q)^2(k'+q)^2(k'-q)^2} \log \frac{(k+k')^2}{(k+q)^2(k'+q)^2(k-q)^2(k'-q)^2} \right. \\ & + \frac{1}{(k+q)^2(k'+q)^2} \log(k+q)^2(k'+q)^2 + \frac{1}{(k-q)^2(k'-q)^2} \log(k-q)^2(k'-q)^2 \Big) \\ & + \frac{2\pi(k+q)^2(k-q)^2(k'+q)^2(k'-q)^2}{(k-k')^2} \\ & \times \left( \frac{1}{(k+q)^2(k'-q)^2} \log(k+q)^2(k'-q)^2 + \frac{1}{(k'+q)^2(k-q)^2} \log(k'+q)^2(k-q)^2 \right. \\ & \left. - \frac{(k'^2 - q^2)(k^2 - q^2) - 2(qk')(qk)}{(k+q)^2(k-q)^2(k'+q)^2(k'-q)^2} \log \frac{(k'+q)^2(k-q)^2(k+q)^2(k'-q)^2}{(k-k')^2} \right) \\ & + \frac{2\pi}{(k+k')^2} \log(k+k')^2(k+q)^2(k'+q)^2 + \frac{2\pi}{(k+k')^2} \log(k'-q)^2(k+q)^2(k'+q)^2 \\ & + \frac{2\pi}{(k+k')^2} \log(k+q)^2(k'-q)^2(k-q)^2. \end{aligned} \quad (5.5)$$

## 6. THE NON-FORWARD EXTENSION OF $\mathcal{K}_2$

From the discussion of the last Section, it is clear that to find an extension

of  $\mathcal{K}_2$  that satisfies the Ward identity constraints, we must weaken the thresholds in  $\mathcal{I}_{AB}(q)$  at  $q \pm k \rightarrow 0$  and  $q \pm k' \rightarrow 0$ . A simple way to achieve this is to remove the denominator in (3.4). To retain the correct dimension we must modify the “vertex function” and reduce the degree of the zeroes at  $q \pm k \rightarrow 0$ ,  $q \pm k' \rightarrow 0$ . Consequently we now define

$$\mathcal{K}_2(k, k', q) = (q^2 - k^2)(q^2 - k'^2) \int dl dl^* [A^* B + AB^*] \quad (6.1)$$

Using extensively the first integral in (3.6) we obtain

$$\mathcal{K}_2(k, k', q) = \left( \frac{(k^2 - k'^2)(q^2 - k^2)(q^2 - k'^2)}{(k + k')^2(k - k')^2} \right) \log \left[ \frac{(q + k)^2(q - k)^2}{(q + k')^2(q - k')^2} \right] \quad (6.2)$$

Clearly

$$\mathcal{K}_2(q, k, k') \xrightarrow{q^2 \rightarrow 0} \mathcal{K}_2 \quad (6.3)$$

It is also manifest that  $\mathcal{K}_2(k, k', q) \rightarrow 0$  when  $q \pm k \rightarrow 0$  or  $q \pm k' \rightarrow 0$ . Consequently  $\mathcal{K}_2(k, k', q)$  satisfies the Ward identity constraints, has all the right symmetries, and has singularities only at the desired thresholds.

## 7. CONCLUSIONS

It is clearly of considerable interest to study the conformal properties of  $\mathcal{K}_2(q, k, k')$  in the conjugate impact parameter space. Given the parallel with the leading-order kernel it is natural to expect that we will find an analogous conformal invariance property. Indeed we conjecture that the Ward Identity constraints are the crucial feature that determine the conformally invariant non-forward extension of a kernel with a holomorphically factorizable spectrum.

It is interesting that to obtain  $\mathcal{K}_2(k, k', q)$  we had to abandon the transverse momentum integral formalism and go to the complex momentum formalism of Kirschner. This is clearly related to the natural connection between the complex momenta formalism and conformal symmetry. It is also consistent with the limitations of the transverse momentum integral formalism uncovered in the unitarity analysis of [5].

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## Appendix A. Regularization of Integrals

This appendix illustrates in more detail the procedure adopted in the regularization of the various integrals we encounter. As an example let's consider

$$I_1 = \int_{\Lambda_1} \frac{d^2 l}{l(l + \eta)^*} \quad (\text{A.1})$$

in which the complex integration region is defined for  $|l| < \Lambda_1$ , since there is an UV divergence. We rewrite it as a contour integral on the unit circle

$$I_1 = \int_0^{\Lambda_1} d|l| \oint \frac{dw}{iw(1 + w \frac{\eta^*}{|l|})} \quad (\text{A.2})$$

and perform the contour integral to get

$$I_1 = 2\pi \int_{|\eta|}^{\Lambda_1} d|l| = 2\pi \log \frac{\Lambda_1}{|\eta|} \quad (\text{A.3})$$

Complex changes of variables are also allowed

$$\begin{aligned} k &\rightarrow l + \eta = l' \\ l^* &\rightarrow l^* + \eta^* = l'^* \\ dldl^* &= dl'dl'^* \end{aligned} \quad (\text{A.4})$$

giving

$$\begin{aligned} \int_{\Lambda_1 \mu_1} \frac{dldl^*}{|l + \eta|^2} &= 2\pi \int_{\mu_1}^{\Lambda_1} \frac{dl'dl'^*}{|l'|^2} \\ &= 2\pi \log \frac{\Lambda_1}{\mu_1}. \end{aligned} \quad (\text{A.5})$$

Notice that this integral is a “massless tadpole” and, as we know, in dimensional regularization (DR) it is hard to make sense out of it both in  $2 + \epsilon$  and in  $2 - \epsilon$



dimensions. Therefore, massless tadpoles, in DR are set to be zero. This is not the case in our analysis and eq. (A.5), therefore, has to be handled with a special care.

In order to further illustrate the last point let's consider the integral

$$b = \int \frac{d^2 l}{|l - \eta|^2} = \frac{1}{i\eta^*} \int_0^{\Lambda_1} d|l| \oint \frac{dw}{(w - \frac{\eta}{|l|})(\frac{|l|}{\eta^*} - w)} \quad (\text{A.6})$$

where we have again rewritten the angular integral in a contour form. The radial integral is ill defined. In our case we get

$$b = 2\pi \int_{|\eta|}^{\infty} \frac{d|l||l|}{|l|^2 - |\eta|^2} - 4\pi \int_0^{|\eta|} \frac{d|l||l|}{|l|^2 - |\eta|^2} \quad (\text{A.7})$$

A careful evaluation then gives

$$\begin{aligned} b &= \pi \log \frac{4(\Lambda_1^2 - |\eta|^2)}{\mu_1^2} \\ &= 2\pi \log \frac{\Lambda_1}{\mu_1} + (2\pi \log 2). \end{aligned} \quad (\text{A.8})$$

Notice that this last term  $(2\pi \log 2)$  has to be omitted in order to obtain consistent results. By doing so the result is consistent with dimensional regularization.

The method, therefore, simply consists of a combination of partial fractioning of complex propagators with an application of eq. (3.6) to evaluate all the integrals involved. Partial fractioning introduces spurious singularities at intermediate stages, which cancel only if the integral is well defined. For example, let's consider

$$\begin{aligned} \int_{\mu_1} \frac{dldl^*}{|l(l+k+k')|^2} &= \int \frac{dldl^*}{|k+k'|^2} \left| \frac{1}{l} - \frac{1}{l+k+k'} \right|^2 \\ &= \frac{2\pi}{|k+k'|^2} \left( \log \frac{\Lambda_1}{\mu_1} - \log \frac{\Lambda_1}{|k+k'|} \right) \\ &= \frac{2\pi}{|k+k'|^2} \log \frac{|k+k'|}{\mu_1}, \end{aligned} \quad (\text{A.9})$$

where the  $\text{Log} \Lambda_1$  terms cancel in the final result. It can be shown that all such terms disappear in the final expression of the box, as expected. Notice that (A.9) is exactly what one expects from dimensional regularization after expanding the result for the

self energy diagram in  $D = 2 + \epsilon$  dimensions and introducing a renormalization scale  $\mu_1$ . In fact

$$\int \frac{d^{2+\epsilon}}{l^2(l+k+k')^2} = \frac{\pi}{(k+k')^2} \text{Log} \frac{(k+k')^2}{\mu_1^2} + O(\epsilon). \quad (\text{A.10})$$

The derivation of  $I_1$ , given in section 3, proceeds as follows. After a partial fractioning we get

$$I_1 = \frac{1}{\eta(k+q-\eta)^*} \int dldl^* \left( \frac{1}{l(l+\eta)^*} - \frac{1}{l(l+k+q)^*} - \frac{1}{|k+\eta|^2} + \frac{1}{(l+\eta)(l+k+q)^*} \right) \quad (\text{A.11})$$

Using (3.6) we get

$$I_1 = \frac{2\pi}{\eta(k+q-\eta)^*} \left( \log \frac{\Lambda_1}{|\eta|} - \log \frac{\Lambda_1}{|k+q|} - \log \frac{\Lambda_1}{\mu_1} + \log \frac{\Lambda_1}{|k+q-\eta|} \right). \quad (\text{A.12})$$

Combining all the 4 terms together we get the result given in section 3.

As another example we consider

$$I_2 = \int \frac{d^2l}{|l+\eta|^2(l+k+q)^*(l+k'+q)} \quad (\text{A.13})$$

After partial fractioning the denominator

$$\begin{aligned} \frac{1}{(l+\eta)(l+\eta)^*(l+k+q)^*(l+k'+q)} &= \frac{1}{(k+q-\eta)(k'+q-\eta)^*} \\ &\left( \frac{1}{|l+\eta|^2} - \frac{1}{(l+\eta)(l+k'+q)^*} \right. \\ &\left. - \frac{1}{l+k+q)(l+\eta)^*} + \frac{1}{(l+k+q)((l+k'+q))} \right) \end{aligned} \quad (\text{A.14})$$

and closing contour integrations in the various sub-integrals we get

$$\begin{aligned} I_2 &= \frac{2\pi}{(k+q-\eta)(k'+q-\eta)^*} \left( \log \frac{\Lambda}{\mu_1} - \frac{1}{|k'+q-\eta|} \log \frac{\Lambda}{|k'+q-\eta|} \right. \\ &\left. - \frac{1}{|k+q-\eta|} \log \frac{\Lambda}{|k+q-\eta|} + \frac{1}{|k'-k|} \log \frac{\Lambda}{|k'-k|} \right) \\ &= \frac{2\pi}{(k+q-\eta)(k'+q-\eta)^*} \log \frac{|k+q||k-q||k'+q||k'-q|}{4|q|^2\lambda|k-k'|} \end{aligned} \quad (\text{A.15})$$

An additional check on the consistency of the methods of regularization, after partial fractioning, comes from the cancellation of the  $q^2 \log q^2$  terms in the non-forward kernel. We have briefly mentioned this important point in Section 5. Since the proof is not obvious, we briefly sketch it here. We introduce a new subtraction, denoted as  $R_q$ , which isolates the  $\log 4q^2$  terms in the “mixed” contributions. After some appropriate manipulations we get

$$R_q I[AB^* + c.c.] = -\frac{RR' [(k+q-\eta)^*(k'+q-\eta)\eta(k+k'-\eta) + c.c.]}{4|q|^2|k+q-\eta|^2|k'+q-\eta|^2|\eta|^2|k+k'-\eta|^2} = 8|q|^2, \quad (\text{A.16})$$

(an overall factor of  $2\pi$  has been omitted). Similarly

$$R_q * I[|A|^2] = \frac{RR'}{4|q|^2|k+k'|^2} \left( -\frac{1}{\eta(k+k'-\eta)^*} - \frac{1}{\eta^*(k+k'-\eta)} - \frac{1}{|\eta|^2} - \frac{1}{|k+k'-\eta|^2} \right) = -4|q|^2 \quad (\text{A.17})$$

and

$$R_q * I[|B|^2] = \frac{RR'}{4|q|^2|k-k'|^2} \left( -\frac{1}{|k+q-\eta|^2} - \frac{1}{|k'+q-\eta|^2} + \frac{1}{(k+q-\eta)(k'+q-\eta)^*} + \frac{1}{(k+q-\eta)^*(k'+q-\eta)} \right) = -4|q|^2. \quad (\text{A.18})$$

Therefore we have the identity

$$R_q * I[box] = R_q \left( I[AB^* + c.c.] + I[|A|^2] + I[|B|^2] \right) = 0 \quad (\text{A.19})$$

as should be the case.

The expression for  $(1 - R_q - R_{\mu_1}) * I[AB^* + c.c.]$  has been given in Section 5. Using this, together with the identity

$$I[box] = (1 - R_q) * I[AB^* + c.c.] + (1 - R_q) * (I[|A|^2] + I[|B|^2]). \quad (\text{A.20})$$

we obtain for the other parts of the box

$$(1 - R_q - R_{\mu_1}) * I[|A|^2] = \frac{2\pi RR'}{|k+k'|^2}$$

$$\begin{aligned}
& \times \left( \frac{(|k'|^2 - |q|^2)(|k|^2 - |q|^2) + [qk][qk']}{RR'} \log \frac{|k + k'|^2}{RR'} \right. \\
& \left. + \frac{2}{|k + q|^2 |k' + q|^2} \log |k + q|^2 |k' + q|^2 + \frac{2}{|k - q|^2 |k' - q|^2} \log |k - q|^2 |k' - q|^2 \right),
\end{aligned} \tag{A.21}$$

and

$$\begin{aligned}
(1 - R_q - R_{\mu_1}) * I[|B|^2] &= \frac{2\pi RR'}{|k - k'|^2} \\
& \times \left( \frac{(|k'|^2 - |q|^2)(|k|^2 - |q|^2) - [qk][qk']}{RR'} \log \frac{|k - k'|^2}{RR'} \right. \\
& \left. + \frac{2}{|k + q|^2 |k' - q|^2} \log |k + q|^2 |k' - q|^2 + \frac{2}{|k - q|^2 |k' + q|^2} \log |k - q|^2 |k' + q|^2 \right).
\end{aligned} \tag{A.22}$$

Using these relations and the identity (4.7) we obtain an expression for the full NLO connected kernel of the form

$$\begin{aligned}
K_2^{(4)} + K_3^{(4)} + K_4^{(4)} &= K_2^{(4)} + K_3^{(4)} + I[box] \\
&= (1 - R_q - R_{\mu_1}) * (K_2^{(4)} + K_3^{(4)}) + (1 - R_q - R_{\mu_1}) * I[AB^* + \\
&+ (1 - R_q - R_{\mu_1}) * (I[|A|^2] + I[|B|^2])].
\end{aligned} \tag{A.23}$$

## Appendix B. Spectrum Evaluation.

This appendix contains some comparison of two possible ways to evaluate the spectrum of the various kernel components that we have discussed. The first method, already presented in [7], is based on the use of dimensional regularization. The second method has been briefly discussed by Kirschner[6]. In the case of the forward kernel, the evaluation of the spectrum of the new partial wave component is more easily performed by using dimensional regularization. However, we anticipate that the second approach may be useful for studying the non-forward kernel and so we give more details here.

A simple treatment of the bubble diagram or 2-point function,  $J_1(k)$ , illustrates both methods. In the approach of [6] we work in  $D = 2$  and regulate the 2-point

function by a cutoff ( $\lambda$ ) using the integral representation

$$\theta[|k' - k| - \lambda] = \frac{1}{2\pi i} \int_{-i\infty+\delta}^{i\infty+\delta} \frac{d\omega}{\omega} \left[ \frac{|k' - k|^2}{\lambda^2} \right]^\omega. \quad (\text{B.1})$$

The contour in  $\omega$  is closed on the right half plane or on the left half plane depending on whether  $|k' - k| < \lambda^2$  or  $|k' - k| > \lambda^2$  respectively. All the diagrams will now depend on  $\omega$  and the final result in the calculation of the spectrum is obtained by extracting the residui for  $\omega = 0$  of the eigenvalues (i.e. performing the integral over  $\omega$  at the end).

This method allows us to work directly at  $D = 2$  without the need of a mass cutoff in the propagators. Therefore, we regulate  $J_1(k)$  by

$$J_{1,reg}(k) = \frac{1}{(2\pi i)^2} \int \frac{d\omega d\omega_0}{\omega \omega_0} \int \frac{d^2 k'}{|k'|^2 |k - k'|^2} \theta[|k' - k| - \lambda] \theta[|k'| - \lambda]. \quad (\text{B.2})$$

(For simplicity we omit an overall factor  $1/(16\pi)^3$  compared to (2.5)). After inserting the representation of the step-function given by (B.1) and after performing the integral over the loop momentum with the help of the formula (with  $D = 2$ )

$$\begin{aligned} I[R, S] &= \int \frac{d^D k}{[(k - q)^2]^R [k^2]^S} \\ &= \frac{\pi^{D/2} \Gamma[D/2 - R] \Gamma[D/2 - S] \Gamma[R + S - D/2]}{\Gamma[R] \Gamma[S] \Gamma[D - R - S] [q^2]^{R+S-D/2}}, \end{aligned} \quad (\text{B.3})$$

it is straightforward to obtain

$$J_{1,reg}(k) = \frac{1}{(2\pi i)^2 k^2} \int \frac{d\omega d\omega_0}{\omega \omega_0} \left( \frac{k^2}{\lambda^2} \right)^{\omega+\omega_0} C(\omega, \omega_0) \quad (\text{B.4})$$

where

$$C(\omega, \omega_0) = \left( \frac{1}{\omega} + \frac{1}{\omega_0} \right) f(\omega, \omega_0), \quad (\text{B.5})$$

with

$$f(\omega, \omega_0) = \frac{\pi \Gamma[1 - \omega - \omega_0] \Gamma[1 + \omega] \Gamma[1 + \omega_0]}{\Gamma[1 - \omega_0] \Gamma[1 - \omega] \Gamma[1 + \omega + \omega_0]}. \quad (\text{B.6})$$

After expanding in  $\omega$  and  $\omega_0$  and picking up the residui at the single poles in both variables we get

$$\begin{aligned} J_{1,reg}(k) &= \frac{1}{(2\pi i)^2 k^2} \int \frac{d\omega d\omega_0}{\omega \omega_0} (2\pi \text{Log} \frac{k^2}{\lambda^2} + \dots) f(\omega, \omega_0) \\ &= \frac{2\pi}{k^2} \text{Log} \frac{k^2}{\lambda^2}, \end{aligned} \quad (\text{B.7})$$

which is the usual (cutoff) expression. As shown in [7], the eigenfunctions of the NLO unitarity kernel are of the form  $f_{n,\nu}(k) = (|k|^2)^{1/2+i\nu} (\frac{k}{|k|})^n$ , as in the lowest order case. Convoluting  $J_1(k)$  with these eigenfunctions one gets the eigenvalue equation

$$\begin{aligned} J_{1,reg} * f_{n,\nu} &\equiv (k^2)^2 \int \frac{d^2 k'}{|k'|^2 |k|^2} J_{1,reg}(k - k') (|k'|^2)^{1/2+i\nu} \left( \frac{k'}{k} \right)^n \\ &= \lambda_s(a, b) f_{n,\nu}. \end{aligned} \quad (\text{B.8})$$

Notice that in (B.8) we have divided by the factor  $1/(k^2 k'^2)$  which appear in the definition of the convolution product [7], and introduced a vertex factor  $(k^2)^2$ , as discussed in Section 2. The singularity at  $k' = 0$  does not need regularization since it is taken care of by the power behaviour of the eigenfunctions. We get

$$J_{1,reg} * f_{n,\nu} = \frac{k^2}{(2\pi i)^2} \int \frac{d\omega d\omega_0}{\omega \omega_0} \frac{C(\omega, \omega_0) \alpha(k)}{(\lambda^2)^{\omega+\omega_0}}, \quad (\text{B.9})$$

where

$$\alpha(k, a, b) = \int \frac{dk' d\theta |k'|^n e^{in\theta}}{(|k'|^2)^a (|k' - k|^2)^b}. \quad (\text{B.10})$$

with  $a = 1/2 - i\nu + n/2$  and  $b = 1 - \omega - \omega_0$ . The angular integral has branch cuts, due to the  $\omega$  terms in the exponents of the denominator. The same difficulty appears in dimensional regularization.

In order to understand this last issue we illustrate the point in detail. The structure of the angular integral in  $\alpha(k)$  is of the form

$$\begin{aligned} I &= \int_0^{2\pi} \frac{d\theta e^{in\theta}}{(1 - z \cos\theta)^\eta} \\ &= i (-1)^{\eta+1} 2^\eta \oint \frac{dw w^{n-1+\eta}}{(zw^2 - 2w + z)^\eta} \\ &= i 2^\eta \oint \frac{dw (-1)^{\eta+1} w^{n+1-\eta}}{z^\eta (w - w_1)^\eta (w - w_2)^\eta} \end{aligned} \quad (\text{B.11})$$

with cuts between  $[0, w_1 = 1/z(1 - \sqrt{1 - z^2})]$  and  $[w_2 = 1/z(1 + \sqrt{1 - z^2}), \infty]$ . We rewrite it in the form

$$I = i(-1)^{\eta+1}(e^{2\pi i\eta} - 1) \int_0^{w_1} \frac{dw w^{n+1-\eta}}{|w_1 - w|^\eta |w_2 - w|^\eta}, \quad (\text{B.12})$$

where we have used the expression of the discontinuity of the factor in the integrand ( $g(w) = -i/((w_1 - w)^\eta(w_2 - w)^\eta)$ ) on the first interval  $[0, w_1]$

$$\begin{aligned} g^+(w) - g^-(w) &= -i \frac{e^{2\pi i\eta} - 1}{|w_1 - w|^\eta |w - w_2|^\eta} \\ &= 2(-1)^\eta \frac{\pi}{B[\eta, 1 - \eta] |w_1 - w|^\eta |w - w_2|^\eta}, \end{aligned} \quad (\text{B.13})$$

where we have used  $\sin \pi \eta = \pi/B[\eta, 1 - \eta]$  by an analytic continuation, and  $B[x, y]$  is the beta-function.

We now obtain

$$\begin{aligned} I &= \frac{2^{\eta+1}\pi}{B[\eta, 1 - \eta]} \frac{z^\eta w_1^n}{w_2^\eta} \int_0^1 \frac{dx x^{n-1+\eta}}{|x - 1|^\eta ((w_1/w_2)x - 1)^\eta} \\ &= \frac{2^{\eta+1}\pi}{z^\eta n B[\eta, n]} \frac{w_1^n}{w_2^\eta} F_{2,1}[\eta, n + \eta, 1 + n, \frac{w_1}{w_2}]. \end{aligned} \quad (\text{B.14})$$

having used as a definition of the hypergeometric function

$$F_{2,1}[a, b, c, z] = \frac{1}{B[b, c - b]} \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - tz)^{-a} dt \quad (\text{B.15})$$

Along the same lines, following the derivation presented above, one can easily evaluate the integral

$$\begin{aligned} I'[\eta] &= \int_0^{2\pi} \frac{d\theta e^{in\theta}}{(1 + v^2 - 2v \cos \theta)^\eta} \\ &= i \oint \frac{dw (-1)^{\eta+1} w^{n+1-\eta}}{v^\eta (w - w_1)^\eta (w - w_2)^\eta}, \end{aligned} \quad (\text{B.16})$$

where now the cuts are between  $[0, w_1 = v]$  and  $[w_2 = 1/v, \infty]$ . Notice that (B.11) and (B.12) are quite general. Therefore it is a simple exercise to show that

$$I'[\eta] = \frac{2\pi v^n}{nB[\eta, n]} F_{2,1}[\eta, \eta + n, n + 1, v^2]. \quad (\text{B.17})$$

The radial integral in  $k$  can now be done and the result, as we are going to show, can be expressed in terms of simple hypergeometrics or also as a string of products of Gamma-functions. This second form is obtained by using the Schwinger parametrization of the integrand. Notice that in the evaluation of the spectrum of  $\mathcal{K}_2$  these difficulties are not present [7] since the angular integral has just two single poles - the angular integration being embedded in D-dimensions.

In dimensional regularization we define

$$\begin{aligned} J_1 * f_{n,\nu} &\equiv (k^2)^2 \int \frac{d^D k'}{|k'|^2 |k|^2} J_1(k - k') (|k'|^2)^{1/2+i\nu} \left(\frac{k'}{k}\right)^n \\ &= \lambda_s(a, b) f_{n,\nu}. \end{aligned} \quad (\text{B.18})$$

with  $D = 2 + \epsilon$  and embed  $\theta$  in a D-dimensional angular space parameterized by  $(\theta_1, \theta_2, \dots, \theta_{D-1})$  by assuming  $\theta \equiv \theta_{D-1}$ . Using the expression of  $I'[\eta]$  given above it is not hard to show that

$$\begin{aligned} &\int_0^{2\pi} \frac{d\theta e^{in\theta}}{(k^2 + k'^2 - 2kk'\cos(\theta - \chi))^\eta} \\ &= \theta[k' - k] \frac{2\pi e^{in\chi}}{(k'^2)^{2-D/2} nB[\eta, n]} \left(\frac{k'}{k}\right)^n F_{2,1}[\eta, \eta + n, n + 1, (k/k')^2] \\ &+ \theta[k - k'] \frac{2\pi e^{in\chi}}{(k^2)^{2-D/2} nB[\eta, n]} \left(\frac{k}{k'}\right)^n F_{2,1}[\eta, \eta + n, n + 1, (k'/k)^2], \end{aligned} \quad (\text{B.19})$$

with  $\cos \chi = k \cdot \hat{x}$  and  $\cos \theta = k' \cdot \hat{x}$  and  $\eta = 2 - D/2$ . Then we get

$$J_1 * f_{n,\nu} = \frac{2\pi^D \Gamma[D/2 - 1]^2 \Gamma[2 - D/2 + n]}{\Gamma[n + 1] \Gamma[D - 2] \Gamma[D/2]} (\sigma_1 + \sigma_2) (k^2)^{D-2} f_{n,\nu}, \quad (\text{B.20})$$

with

$$\begin{aligned} \sigma_1 &= \int_0^1 dx x^{n+(D-2)+2i\nu} F_{2,1}[\eta, \eta + n, n + 1, x^2] \\ \sigma_2 &= \int_0^1 \frac{dx}{x^{(2D-4)-n+2i\nu}} F[\eta, \eta + n, n + 1, x^2]. \end{aligned} \quad (\text{B.21})$$



Notice that the spurious factors containing  $(k^2)^{D-2}$ , with  $D = 2 + \epsilon$  are eliminated at the end, when all components of the spectrum are combined and the singularities cancel. One easily gets

$$\begin{aligned}\sigma_1 &= \frac{1}{\rho_1} F_{3,2}[\eta, \eta + 1 \mid \rho_1, n + 1, \rho + 1, 1] \\ \sigma_2 &= \frac{1}{\rho_2} F_{3,2}[\eta, \eta + 1, \rho_2 \mid n + 1, \rho_2 + 1, 1],\end{aligned}\tag{B.22}$$

with  $\rho_1 = n/2 - 1/2 + D/2 + i\nu$  and  $\rho_2 = 5/2 - D - 2 - i\nu + n/2$ .

One reason for expecting that the cutoff regularization might turn out to be useful in the investigation of the spectrum of the nonforward kernel is that, at least to leading order, the eigenfunctions are given by conformal partial waves, which are known in  $D = 2$  [9]. To our knowledge, however, a direct calculation of the spectrum using these eigenfunctions has not been attempted, even for the BFKL kernel.

In order to conclude our illustration of the method of calculation of the spectrum for the 2-point function, we reconsider  $\alpha(k, a, b)$ , which we rewrite in exponential form

$$\begin{aligned}\alpha(k, a, b) &= \int_0^1 dw_1 dw_2 \delta(1 - w_1 - w_2) \\ &\times \int d^2 k' \int_0^\infty dx \, x^{a+b-1} e^{-x(\vec{k}' - \vec{k} w_2)^2 + x \vec{k}^2 w_2^2 - x w_2 \vec{k}^2} \frac{w_1^{a-1} w_2^{b-1}}{\Gamma[a-1] \Gamma[b-1]} k'^n.\end{aligned}\tag{B.23}$$

To obtain (B.23) we have used the Schwinger parametrization of the propagators and performed a scaling on the integration parameters by  $x$ .

Notice that we have used a mixed real and complex notation (for instance  $\vec{k} \cdot \vec{k}' = 1/2(k k'^* + k^* k')$ , and so on) for convenience. Notice, in particular, that  $k'^n$  is a complex vector. It is easy to show that

$$\int d^2 k' k'^n e^{-x(k' + k w_2)^2} = \frac{\pi}{x} k^n w_2^n.\tag{B.24}$$

(B.24) is easily derived by a complex change of coordinates in the momentum integration and using the complex expansion  $(k + k')^n = \sum_{p=0}^n n!/(k!(n-k)!) k^p k'^{n-p}$ . Only the  $n = p$  term is nonvanishing after angular integration. The integration over  $x$  is

also gaussian and we get

$$\begin{aligned}\alpha(k, a, b) &= \frac{\pi k^n}{(|k|^2)^{a-1} \Gamma[a-1] \Gamma[b-1]} \Gamma[a+b-1] \left( \frac{k^2}{\lambda^2} \right)^{\omega+\omega_0} \int_0^1 dw_2 (1-w_2)^{-b} w_2^{n-a} \\ &= \lambda_s(a, b) f_{n,\nu},\end{aligned}\tag{B.25}$$

(since  $a-1 = -1/2 + n/2 - i\nu$ ). Notice that the additional factor  $(k^2/\lambda^2)^{\omega+\omega_0}$  is removed after the final integration in the  $\omega$  variables which sets  $\omega = \omega_0 = 0$ . After some manipulations we obtain

$$\lambda_s(a, b) = \pi \Gamma[a+b-1] \Gamma[1-a] \Gamma[1-b] B[1+n-a, 1-b].\tag{B.26}$$

Finally, after performing the integrals over  $\omega, \omega_0$  in the expression above, only the residui at the simple poles in these variables survive. A similar approach can be followed also in dimensional regularization. We conclude by recalling that the observation in [7] that part of the  $O(g^4)$  forward kernel is simply the square of the  $O(g^2)$  Lipatov kernel allowed us to write down immediately the corresponding spectrum.

## References

- [1] E. A. Kuraev, L. N. Lipatov, V. S. Fadin, *Sov. Phys. JETP* **45**, 199 (1977) ;  
Ya. Ya. Balitsky and L. N. Lipatov, *Sov. J. Nucl. Phys.* **28**, 822 (1978).
- [2] V. S. Fadin, presentation at the Gran Sasso QCD Summer Institute (1994);  
V. S. Fadin and L. N. Lipatov, *Nucl. Phys.* **B406**, 259 (1993).
- [3] A. R. White, *Phys. Lett.* **B334**, 87 (1994).
- [4] C. Corianò and A. R. White ANL-HEP-CP-94-79, Proceedings of the XXIV International Symposium on Multiparticle Dynamics, Vietri sul Mare, Italy (1994).
- [5] C. Corianò and A. R. White , ANL-HEP-PR-95-19.
- [6] R. Kirschner, LEIPZIG-18-1995, hep-ph/9505421.
- [7] C. Corianò and A. R. White *Phys. Rev. Lett.* **74**, 4980 (1995); C. Corianò and A. R. White *Nucl. Phys.* **B451**, 231 (1995).
- [8] G. Källen and J. Toll, *J. Math. Phys.* **6**(1965) 299.
- [9] L. N. Lipatov, in *Perturbative QCD*, ed. A. .H. Mueller (World Scientific, 1989).